ALGEBRAIC CURVES
EXERCISE SHEET 8

Unless otherwise specified, k is an algebraically closed field.

Exercise 1.

Let $r \geq 1$, $P \in \mathbb{A}_k^r$. Call $\mathcal{O} := \mathcal{O}_P(\mathbb{A}_k^r)$ and $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$.

(1) Compute $\chi(n) = dim_k(\mathcal{O}/\mathfrak{m}^n)$ for $r = 1, 2$.
(2) For arbitrary r, show that $\chi(n)$ is a polynomial of degree r in n with leading coefficient $1/r!$.

Exercise 2.

Find the multiple points and the tangent lines at the multiple points for each of the following curves:

(1) $X^4 + Y^4 - X^2Y^2$
(2) $X^3 + Y^3 - 3X^2 - 3Y^2 + 3XY + 1$
(3) $Y^2 + (X^2 - 5)(4X^4 - 20X^2 + 25)$

Exercise 3.

Let $T : \mathbb{A}_k^2 \to \mathbb{A}_k^2$ be a polynomial map, $Q \in \mathbb{A}_k^2$ and $P = T(Q)$. If T is written component-wise as (T_1, T_2), the Jacobian matrix of T at Q is defined as $J_Q(T) = (\partial T_i/\partial X_j(Q))_{1 \leq i,j \leq 2}$.

(1) Show that $m_Q(F^T) \geq m_P(F)$.
(2) Show that if $J_Q(T)$ is invertible, then $m_Q(F^T) = m_P(F)$.
(3) Show that the converse of the previous statement is false.

Exercise 4.

Let $n \geq 2$ and $F \in k[X_1, \ldots, X_n]$. Consider $V(F) \subseteq \mathbb{A}_k^n$ the associated hypersurface and $P \in V(F)$.

(1) Define the multiplicity $m_P(F)$ of F at P.
(2) If $m_P(F) = 1$, define the tangent hyperplane of F at P.
(3) Can you define tangent hyperplanes for $F = X^2 + Y^2 - Z^2$ at $P = (0, 0, 0)$?
(4) Assume that F is irreducible. Show that, for $n = 2$ (curves), $V(F)$ has finitely many multiple points. Is this true for $n > 2$?

Exercise 5.

Let $R = k[\epsilon]/(\epsilon^2)$ and $\varphi : R \to k$ the k-algebra homomorphism sending ϵ to 0 (R is often called the ring of *dual numbers*). Let $F \in k[X, Y]$ irreducible, $P \in V(F)$, $\mathfrak{m}_P \subseteq \Gamma(F)$ the corresponding maximal ideal and $\theta_P : \Gamma(F) \to \Gamma(F)/\mathfrak{m}_P \simeq k$ the associated k-algebra homomorphism.

(1) Suppose that P is a simple point. Show that there is a bijection between the tangent line to F at P and $\{\theta \in Hom_{k-alg}(\Gamma(F), R) \mid \varphi \circ \theta = \theta_P\}$.
(2) What happens for multiple points (for instance, $F = Y^2 - X^3$, $P = (0, 0)$)?